THE COMPLEX CROWN FOR HOMOGENEOUS HARMONIC SPACES

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1. **Introduction**

Let X be a simply connected homogeneous harmonic Riemannian space. Then according to [5], Corollary 1.2, X is isometric (up to scaling of the metric) to one of the following spaces:

(i) $\mathbb{R}^n$.
(ii) S^n, $P^k(\mathbb{C})$, $P^l(\mathbb{H})$, or $P^2(\mathbb{O})$, i.e., a compact rankone symmetric space.
(iii) $H^n(\mathbb{R})$, $H^k(\mathbb{C})$, $H^l(\mathbb{H})$, or $H^2(\mathbb{O})$, i.e., a noncompact rankone symmetric space.
(iv) a solvable Lie group $S = A \ltimes N$ where N is of Heisenberg-type and $A \simeq \mathbb{R}^+$ acts on N by anisotropic dilations preserving the grading.

We denote by $\mathcal{L}$ the Laplace-Beltrami operator on X. In case (i), $\mathcal{L}$-eigenfunctions on $\mathbb{R}^n$ extend to holomorphic functions on $\mathbb{C}^n$. Likewise, in case (ii), $\mathcal{L}$-eigenfunction on $X = U/K$ admit holomorphic continuation to the whole affine complexification $X_\mathbb{C} = U_\mathbb{C}/K_\mathbb{C}$ of X. This is no longer true in case (iii). However in this case, and more generally for a noncompact Riemannian symmetric space of any rank $X = G/K$, there exists a G-invariant domain $\mathrm{Cr}(X)$ of $X_\mathbb{C} = G_\mathbb{C}/K_\mathbb{C}$ containing X, the *complex crown*, with the following property ([12], [9]):

every $\mathcal{L}$-eigenfunction on X admits a holomorphic extension to $\mathrm{Cr}(X)$, and this domain is maximal for this property.

The objective of this paper is to obtain analogous theory for the spaces in (iv) above.

We note that all spaces in (iii), except for $H^n(\mathbb{R})$, fall into class (iv) by identifying the symmetric space $X = G/K$ with the NA-part in the Iwasawa decomposition $G = NAK$ of a noncompact simple Lie group G of real rank one, and by suitably scaling the metric.

Our investigations start with a new model of the crown domain for the rankone symmetric spaces X. We describe $\mathrm{Cr}(X)$ in terms of the Iwasawa coordinates A and N only; henceforth we refer to this new model as the *mixed model* of the crown. In Section 2 we provide the mixed model for the two basic cases, i.e. the symmetric spaces associated with the groups $G = \mathrm{Sl}(2, \mathbb{R})$ and $G = \mathrm{SU}(2, 1)$.

Starting from the two basic cases, reduction of symmetry allows to obtain a mixed model for all rank one symmetric spaces and motivates a definition of $\mathrm{Cr}(S)$ for the remaining spaces in (iv). This is worked out in Section 3.

Finally, in section 4 we use recent results from [9] to prove holomorphic extension of $\mathcal{L}$-eigenfunctions on S to the crown domain $\mathrm{Cr}(S)$ and establish maximality of $\mathrm{Cr}(S)$ with respect to this property.

2. Mixed model for the crown domain

The crown domain can be realized inside the complexification of an Iwasawa AN-group. Goal of this section is to make this explicit for the rank one groups $\mathrm{Sl}(2,\mathbb{R})$ and $\mathrm{SU}(2,1)$.

2.1. Notation for rank one spaces

Let G be a connected semi-simple Lie group of real rank one. We assume that $G \subset G_\mathbb{C}$ where $G_\mathbb{C}$ is the universal complexification of G. We fix an Iwasawa decomposition $G = NAK$ and form the Riemannian symmetric space

$$X = G/K\,.$$

With $K_\mathbb{C} < G_\mathbb{C}$ the universal complexification of K we arrive at a totally real embedding

$$X \hookrightarrow X_\mathbb{C} := G_\mathbb{C}/K_\mathbb{C}, \quad gK \mapsto gK_\mathbb{C}\,.$$

Let $x_0 = K \in X$ be a base-point.

Let $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}$ and $\mathfrak{n}$ be the Lie algebras of G, K, A and N. Let $\Sigma^+ = \Sigma(\mathfrak{a}, \mathfrak{n})$ be the set of positive roots and put

$$\Omega := \{Y \in \mathfrak{a} \mid \forall \alpha \in \Sigma^+ \ |\alpha(Y)| < \pi/2\}\,.$$

Note that Ω is a symmetric interval in $\mathfrak{a} \simeq \mathbb{R}$. The crown domain of X is defined as

$$\mathrm{Cr}(X) := G\exp(i\Omega)\cdot x_0 \subset X_\mathbb{C}\,. \tag{2.1}$$

Let us point out that Ω is invariant under the Weyl group $\mathcal{W} = N_K(A)/Z_K(A) \simeq \mathbb{Z}_2$ and that $\exp(i\Omega)$ consists of elliptic elements in $G_\mathbb{C}$. We will refer to (2.1) as the *elliptic model* of $\mathrm{Cr}(X)$ (see [12] for the basic structure theory in these coordinates).

Let us define a domain Λ in $\mathfrak{n}$ by

$$\Lambda := \{Y \in \mathfrak{n} \mid \exp(iY)\cdot x_0 \subset \mathrm{Cr}(X)\}_0$$

where $\{\cdot\}_0$ refers to the connected component of $\{\cdot\}$ which contains 0.

The set Λ is explicitly determined in [8], Th. 8.11. Further by [8], Th. 8.3:

$$\mathrm{Cr}(X) = G\exp(i\Lambda)\cdot x_0\,. \tag{2.2}$$

We refer to (2.2) as the *unipotent model* of $\mathrm{Cr}(X)$.

Finally let us mention the fact that $\mathrm{Cr}(X) \subset N_{\mathbb{C}}A_{\mathbb{C}}\cdot x_0$ which brings us to the question whether $\mathrm{Cr}(X)$ can be expressed in terms of A, N, Ω and Λ. This is indeed the case and will be considered in the following two subsections for the groups $\mathrm{Sl}(2,\mathbb{R})$ and $\mathrm{SU}(2,1)$.

2.2. **Mixed model for the upper half plane**

Let

$$G = \mathrm{Sl}(2,\mathbb{R}) \quad \text{and} \quad G_{\mathbb{C}} = \mathrm{Sl}(2,\mathbb{C})\,.$$

Our choices of A, N and K are as follows:

$$A = \left\{ a_t = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \mid t > 0 \right\},$$

$$A_{\mathbb{C}} = \left\{ a_z = \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \mid z \in \mathbb{C}^* \right\},$$

$$K = \mathrm{SO}(2,\mathbb{R}) \quad \text{and} \quad K_{\mathbb{C}} = \mathrm{SO}(2,\mathbb{C})\,,$$

and

$$N = \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\},$$

$$N_{\mathbb{C}} = \left\{ n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

We will identify $X = G/K$ with the upper halfplane $\mathbf{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$ via the map

$$X \to \mathbf{H},\quad gK \mapsto \frac{ai+b}{ci+d} \qquad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right). \tag{2.3}$$

Note that $x_0 = i$ within our identification.

We view $X = \mathbf{H}$ inside of the complex projective space $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ and note that $\mathbb{P}^1(\mathbb{C})$ is homogeneous for $G_{\mathbb{C}}$ with respect to the usual fractional linear action:

$$g(z) = \frac{az+b}{cz+d} \qquad \left(z \in \mathbb{P}^1(\mathbb{C}), g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{C}} \right).$$

We use a more concrete model for $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$, namely

$$X_{\mathbb{C}} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \backslash \mathrm{diag},\quad gK_{\mathbb{C}} \mapsto (g(i), g(-i))$$

which is a $G_{\mathbb{C}}$-equivariant diffeomorphism. With this identification of $X_{\mathbb{C}}$ the embedding of (2.3) becomes

$$X \hookrightarrow X_{\mathbb{C}}, \quad z \mapsto (z, \overline{z}) \,. \tag{2.4}$$

We will denote by $\overline{X}$ the lower half plane and note that the *crown domain* for $\mathrm{Sl}(2,\mathbb{R})$ is given by:

$$\mathrm{Cr}(X) = X \times \overline{X} \,.$$

We note that

$$\Omega = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \mid x \in (-\pi/4, \pi/4) \right\} ,$$

and

$$\Lambda = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \mid x \in (-1,1) \right\} .$$

With that we come to the mixed model for the crown which combines both parameterizations in an unexpected way.

We let $F := \{\pm \mathbf{1}\}$ be the center of G and note that

$$N_{\mathbb{C}} A_{\mathbb{C}} \cdot x_0 = \mathbb{C} \times \mathbb{C} \setminus \mathrm{diag}$$

and

$$N_{\mathbb{C}} A_{\mathbb{C}} \cdot x_0 \simeq N_{\mathbb{C}} A_{\mathbb{C}} / F \,.$$

Proposition 2.1. *Let $G = \mathrm{Sl}(2,\mathbb{R})$. Then the map*

$$NA \times \Omega \times \Lambda \to \mathrm{Cr}(X), \quad (na, H, Y) \mapsto na \exp(iH) \exp(iY) \cdot x_0$$

is an AN-equivariant diffeomorphism.

Proof. By the facts listed above we only have to show that the map is defined and onto. For that we first note:

$$\exp(i\Lambda) \cdot x_0 = \{((1+t)i, -(1-t)i) \mid t \in (-1,1)\}$$

and thus

$$A \exp(i\Lambda) = i\mathbb{R}^+ \times -i\mathbb{R}^+ \,.$$

Consequently

$$A \exp(i\Omega) \exp(i\Lambda) \cdot x_0 = \{(z,w) \in \mathrm{Cr}(X) \mid \arg(w) = \pi + \arg(z)\}$$

and finally

$$NA \exp(i\Omega) \exp(i\Lambda) \cdot x_0 = \mathrm{Cr}(X)$$

as asserted. □

2.3. **Mixed model for** $\mathrm{SU}(2,1)$

Let $G = \mathrm{SU}(2,1)$. We let G act on $\mathbb{P}^2(\mathbb{C}) = (\mathbb{C}^3 \setminus \{0\}/\sim)$ by projectivized linear transformations. We embed $\mathbb{C}^2$ into $\mathbb{P}^2(\mathbb{C})$ via $z \mapsto [z,1]$. Then G preserves the ball

$$X = \{z \in \mathbb{C}^2 \mid \|z\|_2 < 1\} \simeq G/K$$

with the maximal compact subgroup $K = S(U(2) \times U(1))$ stabilizing the origin $x_0 = 0 \in X$. Note that an element

$$g = \begin{pmatrix} A & u \\ v^t & \alpha \end{pmatrix} \in G$$

with $u, v \in \mathbb{C}^2$ acts on $z \in X$ by

$$g(z) = \frac{Az + u}{v^t \cdot z + \alpha}\,.$$

Now, as X is Hermitian, the crown is given by the double

$$\mathrm{Cr}(X) = X \times X$$

but with X embedded in $\mathrm{Cr}(X)$ as $z \mapsto (z, \overline{z})$. We choose

$$A = \left\{ a_t := \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

Set $Y_t := \log a_t$ and note that

$$\Omega = \{Y_t \mid -\pi/4 < t < \pi/4\}\,.$$

According to [8], Th. 8.11 we have

$$\Lambda = \Big\{ Z_{a,b} := \begin{pmatrix} ib & a & -ib \\ -\overline{a} & 0 & \overline{a} \\ ib & a & -ib \end{pmatrix} \mid a \in \mathbb{C}, b \in \mathbb{R}; \\ |a|^2 + |b| < 1/2 \Big\}\,.$$

We define now a subset $\mathcal{D} \subset \mathfrak{a} \times \mathfrak{n}$ by

$$\mathcal{D} := \{(Y_t, Z_{a,b}) \in \mathfrak{a} \times \mathfrak{n} \mid (1 - 2|a|^2 - 2|b|)\cos(2t) > (1 - \cos(2t))|a|^2\}_0\,.$$

If $\pi_{\mathfrak{a}} : \mathfrak{a} \times \mathfrak{n} \to \mathfrak{a}$ denotes the first coordinate projection and $\pi_{\mathfrak{n}}$ resp. the second, then note the following immediate facts:

- $\mathcal{D} \subset \Omega \times \Lambda$,
- $\pi_{\mathfrak{a}}(\mathcal{D}) = \Omega$,
- $\pi_{\mathfrak{n}}(\mathcal{D}) = \Lambda$.

Proposition 2.2. *Let $G = \mathrm{SU}(2,1)$. Then the map*

$$\Phi : NA \times \mathcal{D} \to \mathrm{Cr}(X), \quad (na,(Y,Z)) \mapsto na\exp(iY)\exp(iZ)\cdot x_0$$

is a diffeomorphism.

Proof. All what we have to show is that the map is defined and onto.

To begin with we show that Φ is defined, i.e. $\operatorname{Im}\Phi \subset \mathrm{Cr}(X)$. Set $n_{a,b} := \exp(iZ_{a,b})$. Let $M = Z_K(A)$. By M-invariance it is no loss of generality to assume that a is real. Then

$$n_{a,b} = \begin{pmatrix} 1-b+a^2/2 & ia & b-a^2/2 \\ -ia & 1 & ia \\ -b+a^2/2 & ia & 1+b-a^2/2 \end{pmatrix}.$$

We have to show that:

$$a_{i\phi}n_{a,b}(0) \in X \quad \textit{and} \quad \overline{a_{i\phi}n_{a,b}(0)} \in X$$

for all $|\phi| < \pi/4$ and $|a|^2 + |b| < 1/2$.

Now

$$n_{a,b}(0) = \frac{1}{1+b-a^2/2}(b-a^2/2, ia).$$

Note that $n_{a,b}(0) \in X$ if and only if

$$(b-a^2/2)^2 + a^2 < (1+b-a^2/2)^2$$

or

$$-2b + 2a^2 < 1$$

which is the defining condition of Λ.

Applying $a_{i\phi}$ we obtain that

$$a_{i\phi}n_{a,b}(0) = \frac{\left(\cos\phi\frac{b-a^2/2}{1+b-a^2/2} + i\sin\phi, \frac{ia}{1+b-a^2/2}\right)}{\cos\phi + i\sin\phi\frac{b-a^2/2}{1+b-a^2/2}}.$$

Hence $a_{i\phi}n_{a,b}(0) \in X$ if and only if

$$\cos^2\phi + \sin^2\phi\frac{(b-a^2/2)^2}{(1+b-a^2/2)^2} > \cos^2\phi\frac{(b-a^2/2)^2}{(1+b-a^2/2)^2} + \\ + \sin^2\phi + \frac{a^2}{(1+b-a^2/2)^2}$$

or, equivalently, after clearing denominators:

$$(1+b-a^2/2)^2\cos^2\phi + (b-a^2/2)^2\sin^2\phi > (b-a^2/2)^2\cos^2\phi + \\ + (1+b-a^2/2)^2\sin^2\phi + a^2.$$

Simplifying further we arrive at:

$$\cos^2\phi + 2(b - a^2/2)\cos^2\phi > \sin^2\phi + 2(b - a^2/2)\sin^2\phi + a^2$$

and equivalently:

$$(1 - 2b - 2a^2)\cos(2\phi) > (1 - \cos 2\phi)a^2\,. \tag{2.5}$$

But this is the defining condition for $\mathcal{D}$.

To see that the map is onto we observe that $\mathrm{Cr}(X) \subset N_{\mathbb{C}}A_{\mathbb{C}} \cdot x_0$. Hence there exist a domain $\mathcal{D}' \subset \mathfrak{a} + \mathfrak{n}$ such that

$$\mathrm{Cr}(X) = NA \cdot \{\exp(iY)\exp(iX) \mid (Y, X) \in \mathcal{D}'\}\,.$$

If $\mathcal{D}'$ is strictly larger then $\mathcal{D}$, then $\mathcal{D}'$ contains a boundary point of $\mathcal{D}$. As

$$\partial\mathcal{D} = \partial\Omega \times \Lambda \amalg \Omega \times \partial\Lambda \amalg \partial\Omega \times \partial\Lambda,$$

we arrive at a contradiction with (2.5). □

3. The crown for homogeneous harmonic spaces

Let S be a simply connected noncompact homogeneous harmonic space. According to [5], Corollary 1.2, there are the following possibilities for S:

(i) $S = \mathbb{R}^n$.

(ii) S is the AN-part of a noncompact simple Lie group G of real rank one.

(iii) $S = A \ltimes N$ with N a nilpotent group of Heisenberg-type and $A \simeq \mathbb{R}$ acting on N by graduation preserving scalings.

The case of $S = \mathbb{R}^n$ we will not consider; groups under (i) are referred to as symmetric solvable harmonic groups. We mention that all spaces in (ii), except for those associated to $G = \mathrm{SO}_o(1, n)$, are of the type in (iii). Most issues of the Lorentz groups $G = \mathrm{SO}_o(1, n)$ readily reduce to $\mathrm{SO}_o(1, 2) \simeq \mathrm{PSl}(2, \mathbb{R})$ where comprehensive treatments are available. In fact for the real hyperbolic spaces $X = H^n(\mathbb{R}) = \mathrm{SO}_o(1, n)/\mathrm{SO}(n)$ we obtain the following result, analogous to Proposition 2.1.

Proposition 3.1. *Let $G = \mathrm{SO}_o(1, n)$ $(n \geq 2)$. Then the map*

$$NA \times \Omega \times \Lambda \to \mathrm{Cr}(X), \quad (na, H, Y) \mapsto na\exp(iH)\exp(iY) \cdot x_0$$

is a diffeomorphism.

We will now focus on type (iii). In the sequel we recall some basic facts about H-type groups and their solvable harmonic extensions. We

refer to [13] for a more comprehensive treatment and references. After that we introduce the crown domain for such harmonic extensions.

3.1. H-type Lie algebras and groups

Let $\mathfrak{n}$ be a real nilpotent Lie algebra of step two (that is, $[\mathfrak{n},\mathfrak{n}] \neq \{0\}$ and $[\mathfrak{n},[\mathfrak{n},\mathfrak{n}]] = \{0\}$), equipped with an inner product $\langle\cdot,\cdot\rangle$ and associated norm $|\cdot|$. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement in $\mathfrak{n}$. Then

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, \quad [\mathfrak{v},\mathfrak{z}] = 0, \quad [\mathfrak{n},\mathfrak{n}] = [\mathfrak{v},\mathfrak{v}] \subset \mathfrak{z}.$$

For $Z \in \mathfrak{z}$ let $J_Z : \mathfrak{v} \to \mathfrak{v}$ be the linear map defined by

$$\langle J_Z V, V'\rangle = \langle Z, [V,V']\rangle, \quad \forall V, V' \in \mathfrak{v}.$$

Then $\mathfrak{n}$ is called a *Heisenberg type* algebra (or *H-type* algebra, for short) if

$$J_Z^2 = -|Z|^2 \operatorname{id}_{\mathfrak{v}} \qquad (Z \in \mathfrak{z}). \tag{3.1}$$

A connected and simply connected Lie group N is called an H-type group if its Lie algebra $\mathfrak{n} = \mathrm{Lie}(N)$ is an H-type algebra, see [7].

We let $p = \dim \mathfrak{v}$, $q = \dim \mathfrak{z} (\geq 1)$. Condition (3.1) implies that p is even and $[\mathfrak{v},\mathfrak{v}] = \mathfrak{z}$.

Moreover, (3.1) implies that the map $Z \to J_Z$ extends to a representation of the real Clifford algebra $\mathrm{Cl}(\mathfrak{z}) \cong \mathrm{Cl}_q$ on $\mathfrak{v}$. This procedure can be reversed and yields a general method for constructing H-type algebras.

Since $\mathfrak{n}$ is nilpotent, the exponential map $\exp : \mathfrak{n} \mapsto N$ is a diffeomorphism. The Campbell-Hausdorff formula implies the following product law in N:

$$\exp X \cdot \exp X' = \exp\left(X + X' + \tfrac{1}{2}[X,X']\right), \quad \forall X, X' \in \mathfrak{n}.$$

This is sometimes written as

$$(V,Z)\cdot(V',Z') = \left(V+V', Z+Z' + \tfrac{1}{2}[V,V']\right),$$

using the exponential chart to parametrize the elements $n = \exp(V+Z)$ by the couples $(V,Z) \in \mathfrak{v} \oplus \mathfrak{z} = \mathfrak{n}$.

3.1.1. *Reduction theory.* We conclude this section with reduction theory for H-type Lie algebras to Heisenberg algebras.

Let $\mathfrak{z}_1 = \mathbb{R}Z_1$ be a one-dimensional subspace of $\mathfrak{z}$ and $\mathfrak{z}_1^\perp$ its orthogonal complement in $\mathfrak{z}$. We assume that $|Z_1| = 1$ and set $J_1 := J_{Z_1}$. We form the quotient algebra

$$\mathfrak{n}_1 := \mathfrak{n}/\mathfrak{z}_1^\perp$$

and record that $\mathfrak{n}_1$ is two-step nilpotent. Let $p_1 : \mathfrak{z} \to \mathfrak{z}_1$ be the orthogonal projection. If we identify $\mathfrak{n}_1$ with the vector space $\mathfrak{v} \oplus \mathfrak{z}_1$ via the linear map

$$\mathfrak{n}_1 \to \mathfrak{v} \oplus \mathfrak{z}_1, \quad (V, Z) + \mathfrak{z}_1^\perp \mapsto (V, p_1(Z)),$$

then the bracket in $\mathfrak{n}_1$ becomes in the new coordinates

$$[(V, cZ_1), (V', c'Z_1)] = (0, \langle Z_1, [V, V'] \rangle Z_1)$$

where $c, c' \in \mathbb{R}$ and $V, V' \in \mathfrak{v}$. Since $\langle Z_1, [V, V'] \rangle = \langle J_1 V, V' \rangle$ we see that J_1 determines a Lie algebra automorphism of $\mathfrak{n}_1$ and thus $\mathfrak{n}_1$ is isomorphic to the $p + 1$-dimensional Heisenberg algebra.

3.2. **Harmonic solvable extensions of H-type groups**

Let $\mathfrak{n}$ be an H-type algebra with associated H-type group N. Let $\mathfrak{a}$ be a one-dimensional Lie algebra with an inner product. Write $\mathfrak{a} = \mathbb{R}H$, where H is a unit vector in $\mathfrak{a}$. Let $A = \exp \mathfrak{a}$ be a one-dimensional Lie group with Lie algebra $\mathfrak{a}$ and isomorphic to $\mathbb{R}^+$ (the multiplicative group of positive real numbers). Let the elements $a_t = \exp(tH) \in A$ act on N by the dilations $(V, Z) \to (e^{t/2}V, e^t Z)$ for $t \in \mathbb{R}$, and let S be the associated semidirect product of N and A:

$$S = NA = N \rtimes A.$$

The action of A on N becomes the inner automorphism

$$a_t \exp(V + Z) a_t^{-1} = \exp\left(e^{t/2}V + e^t Z\right), \tag{3.2}$$

and the product in S is given by

$$\exp(V + Z)a_t \exp(V' + Z')a_{t'} = \exp(V + Z) \exp(e^{t/2}V' + e^t Z')a_{t+t'}.$$

S is a connected and simply connected Lie group with Lie algebra

$$\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathfrak{a}$$

and Lie bracket defined by linearity and the requirement that

$$[H, V] = \frac{1}{2}V, \qquad [H, Z] = Z, \qquad \forall V \in \mathfrak{v}, \ \forall Z \in \mathfrak{z}. \tag{3.3}$$

The map $(V, Z, tH) \to \exp(V + Z) \exp(tH)$ is a diffeomorphism of $\mathfrak{s}$ onto S. If we parametrize the elements $na = \exp(V+Z) \exp(tH) \in NA$ by the triples $(V, Z, t) \in \mathfrak{v} \times \mathfrak{z} \times \mathbb{R}$, then the product law reads

$$(V, Z, t) \cdot (V', Z', t') = \left(V + e^{t/2}V', Z + e^t Z' + \tfrac{1}{2}e^{t/2}[V, V'], t + t'\right),$$

for all $V, V' \in \mathfrak{v}$, $Z, Z' \in \mathfrak{z}$, $t, t' \in \mathbb{R}$. For $n = (V, Z, 0) \in N$ and $a_t = (0, 0, t) \in A$ we consistently get $na_t = (V, Z, t)$.

We extend the inner products on $\mathfrak{n}$ and $\mathfrak{a}$ to an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{s}$ by linearity and the requirement that $\mathfrak{n}$ be orthogonal to $\mathfrak{a}$. The left-invariant Riemannian metric on S defined by this inner product turns S into a harmonic solvable group [4].

3.3. **The complexification and the crown**

Let $N_\mathbb{C}$ be the simply connected Lie group with Lie algebra $\mathfrak{n}_\mathbb{C}$ and set $A_\mathbb{C} = \mathbb{C}^*$. Then $A_\mathbb{C}$ acts on $N_\mathbb{C}$ by $t \cdot (V, Z) := (tV, t^2V)$ for $t \in \mathbb{C}^*$ and $(V, Z) \in N_\mathbb{C}$.

We define a complexification $S_\mathbb{C}$ of S by

$$S_\mathbb{C} = N_\mathbb{C} \rtimes A_\mathbb{C}\,.$$

For $z \in \mathbb{C}$ we often set $a_z := \exp(zH)$ and note that a_z corresponds to $e^{z/2} \in \mathbb{C}^*$. In particular

$$\{z \in \mathbb{C} :\ \ \exp(zH) = e\} \subset 4\pi i\mathbb{Z}. \tag{3.4}$$

It follows that the exponential map $\exp : \mathfrak{a}_\mathbb{C} \to A_\mathbb{C}$ is certainly injective if restricted to $\mathfrak{a} \oplus iH(-2\pi, 2\pi]$.

Motivated by our discussion of $\mathrm{SU}(2,1)$ and the discussed $\mathrm{SU}(n,1)$-reduction, we define the following sets for a more general harmonic AN-group.

$$\Omega = \{tH \in \mathfrak{a} :\ \ |t| < \tfrac{\pi}{2}\}, \tag{3.5}$$

$$\Lambda = \{(V, Z) \in \mathfrak{n} :\ \ \tfrac{1}{2}|V|^2 + |Z| < 1\}, \tag{3.6}$$

$$\mathcal{D} = \big\{(V, Z, t) \in \mathfrak{s} :\ \cos t(1 - \tfrac{1}{2}|V|^2 - |Z|) > \tfrac{1}{4}(1 - \cos t)|V|^2\big\}_0\,, \tag{3.7}$$

$$D = \{\exp(itH)\exp(iV + iZ) :\ \ (V, Z, t) \in \mathcal{D}\} \subset S_\mathbb{C}. \tag{3.8}$$

Here as usual $\{\cdot\}_0$ denotes the connected component of $\{\cdot\}$ containing 0, and we write (V, Z, t) for the element $V + Z + tH$ of $\mathfrak{s}$.

The following result is immediate from the definition of $\mathcal{D}$.

Lemma 3.2. *Let $\pi_\mathfrak{n}$ (resp. $\pi_\mathfrak{a}$) denote the projection onto the first (resp. second) factor in $\mathfrak{s} = \mathfrak{n} \times \mathfrak{a}$. Then*

- $\mathcal{D} \subset \Lambda \times \Omega$,
- $\pi_\mathfrak{n}(\mathcal{D}) = \Lambda$,
- $\pi_\mathfrak{a}(\mathcal{D}) = \Omega$.

The set $\mathcal{D}$ is the interior of the closed hypersurface in $\mathfrak{s}$ defined by the equation

$$\cos t(1 - \tfrac{1}{2}|V|^2 - |Z|) = \tfrac{1}{4}(1 - \cos t)|V|^2 \quad (-\tfrac{\pi}{2} \leq t \leq \tfrac{\pi}{2}). \tag{3.9}$$

This can be rewritten as $\cos t(1 - |V|^2/4 - |Z|) = |V|^2/4$, or also as

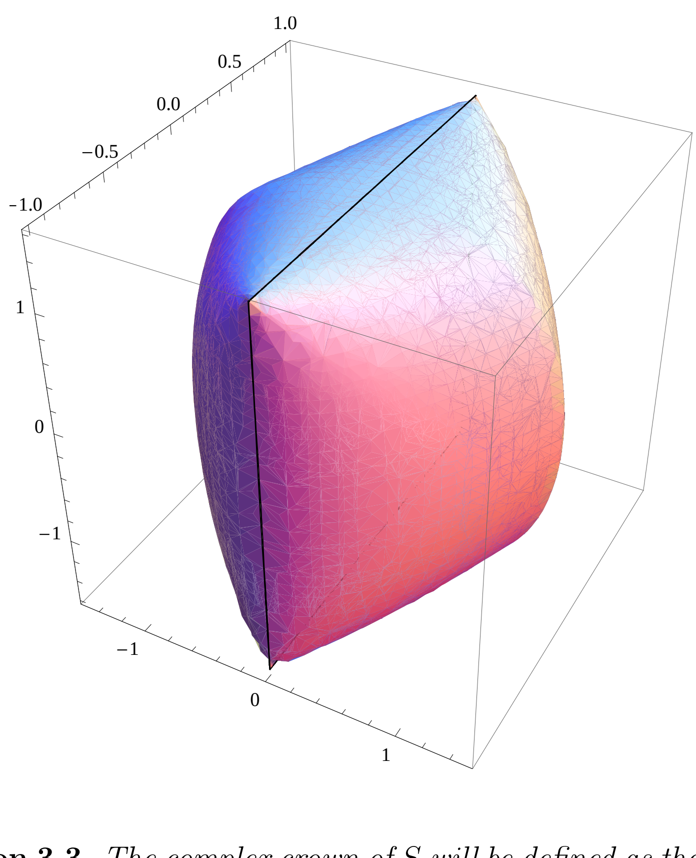
$$1 - \tan^2 \tfrac{t}{2} = \tfrac{|V|^2}{2(1-|Z|)},$$

i.e.,

$$|t| = 2 \arctan \sqrt{1 - \tfrac{|V|^2}{2(1-|Z|)}}.$$

A picture of this hypersurface in $\mathbb{R}^3 = \{(V, Z, t)\}$ for the case $\mathfrak{z} \simeq \mathbb{R}$, $\mathfrak{v} \simeq \mathbb{R}^2$ (with one coordinate suppressed), i.e., for $\mathrm{SU}(2,1)$, is given below. Here $|t| \leq \pi/2$, $|Z| \leq 1$, and $|V| \leq \sqrt{2}$

A plot of the surface (3.9) in $\mathbb{R}^3 = \{(V, Z, t)\}$.



Definition 3.3. *The complex crown of S will be defined as the following subset of $S_{\mathbb{C}}$:*

$$\mathrm{Cr}(S) = NAD \subset NA \exp(i\Omega) \exp(i\Lambda) \subset S_{\mathbb{C}}. \tag{3.10}$$

It is easy to see that $\mathrm{Cr}(S)$ is open and simply connected in $S_{\mathbb{C}}$.

We conclude with the proof of the mixed model of the crown for rank one symmetric spaces.

Theorem 3.4. *Let $X = G/K$ be a non-compact Riemannian symmetric space of rank one, $X \neq H^n(\mathbb{R})$. Then, with $S = NA$,*

$$\mathrm{Cr}(X) = \mathrm{Cr}(S).$$

Proof. We first show that $\mathrm{Cr}(S) \subset \mathrm{Cr}(X)$. We have to show that $\exp(itH)\exp(iY)\cdot x_0 \subset \mathrm{Cr}(X)$ for all $(Y,t) \in \mathcal{D}$. We apply $M = Z_K(A)$ and the assertion is reduced to $G = \mathrm{SU}(2,1)$, where it was shown in Proposition 2.2 above.

In order to conclude the proof of the theorem, it is enough to show that the elements which are in the boundary of D do not lie in $\mathrm{Cr}(X)$, i.e.,

$$\{\exp(itH)\exp(iV + iZ) : \ (V, Z, t) \in \partial\mathcal{D}\} \cap \mathrm{Cr}(X) = \emptyset.$$

This again reduces to $G = \mathrm{SU}(2,1)$, and is easily verified. □

4. Geometric Analysis

In this section we will show that eigenfunctions of the Laplace-Beltrami-Operator on S extend holomorphically to $\mathrm{Cr}(S)$ and that $\mathrm{Cr}(S)$ is maximal with respect to this property.

4.1. Holomorphic extension of eigenfunctions

For $z \in D$ we consider the following totally real embedding of S into $\mathrm{Cr}(S)$ given by

$$S \hookrightarrow \mathrm{Cr}(S), \quad s \mapsto sz\,.$$

Now Let $\mathcal{L}$ be the Laplace-Beltrami operator on S, explicitly given by, see [13]

$$\mathcal{L} := \sum_{j=1}^{p} V_j^2 + \sum_{i=1}^{q} Z_i^2 + H^2 - 2\rho H$$

where the $(V_j)_j$ and $(Z_i)_i$ form an orthonormal basis of $\mathfrak{v}$ and $\mathfrak{z}$ respectively. Here we consider elements $X \in \mathfrak{s}$ as left-invariant vector fields on S. Hence it is clear that $\mathcal{L}$ extends to a left $S_{\mathbb{C}}$-invariant holomorphic differential operator on $S_{\mathbb{C}}$ which we denote by $\mathcal{L}_{\mathbb{C}}$. Now if $M \subset S_{\mathbb{C}}$ is a totally real analytic submanifold, then we can restrict

$\mathcal{L}_{\mathbb{C}}$ to M, in symbols $\mathcal{L}_M$, in the following way: if f is a real analytic function near $m \in M$ and $f_{\mathbb{C}}$ a holomorphic extension of f in a complex neighborhood of m in $S_{\mathbb{C}}$, then set:

$$(\mathcal{L}_M f)(m) := (\mathcal{L}_{\mathbb{C}} f_{\mathbb{C}})(m) .$$

Then:

Proposition 4.1. *For all $z \in D$ the restriction $\mathcal{L}_{Sz}$ is elliptic.*

Proof. To illustrate what is going on we first give a proof for those S related to $G = \mathrm{Sl}(2, \mathbb{R})$. Here we have $D := \exp(i\Omega)\exp(i\Lambda)$ and $\mathcal{L} = V^2 + H^2 - \frac{1}{2}H$. Let $z = \exp(itH)\exp(ixV) \in D$. Using $[H, V] = V$ we get

$$\begin{aligned} \mathrm{Ad}(z)H &= H - ie^{it}xV \\ \mathrm{Ad}(z)V &= e^{it}V . \end{aligned}$$

It follows that the leading symbol, or principal part, of $\mathcal{L}_{Sz}$ is given by:

$$[\mathcal{L}_{Sz}]_{\mathrm{prin}} = (H - ie^{it}xV)^2 + e^{2it}V^2 .$$

Let us verify that $\mathcal{L}_{Sz}$ elliptic. The associated quadratic form is given by the matrix

$$L(z) := \begin{pmatrix} 1 & -ixe^{it} \\ -ixe^{it} & e^{2it}(1 - x^2) \end{pmatrix} .$$

We have to show that $\langle L(z)\xi, \xi\rangle = 0$ has no solution for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \backslash \{0\}$. We look at

$$\xi_1^2 - 2ixe^{it}\xi_1\xi_2 + e^{2it}(1 - x^2)\xi_2^2 = 0 .$$

Now $\xi_2 = 0$ is readily excluded and we remain with the quadric

$$\xi^2 - 2ixe^{it}\xi + e^{2it}(1 - x^2) = 0$$

whose solutions are

$$\begin{aligned} \lambda_{1,2} &= ixe^{it} \pm \sqrt{-x^2e^{2it} - (1 - x^2)e^{2it}} \\ &= ie^{it}(x \pm 1). \end{aligned}$$

These are never real in the domain $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $-1 < x < 1$. The same proof works for those S related to $\mathrm{SO}_o(1, n)$, $n \geq 2$.

To put the computation above in a more abstract framework: it is to show here that for $z \in D$ the operator $L(z) : \mathfrak{s}_{\mathbb{C}} \to \mathfrak{s}_{\mathbb{C}}$ defined by

$$L(z) := \mathrm{Ad}(z)^t \,\mathrm{Ad}(z) \tag{4.1}$$

is elliptic in the sense that $\langle L(z)\xi, \xi\rangle = 0$ for $\xi \in \mathfrak{s}$ implies $\xi = 0$.

So for the sequel we assume that $\mathfrak{n}$ is H-type, i.e. $\mathfrak{z} \neq \{0\}$. We will reduce to the case where N is a Heisenberg group, i.e. S is related to $\mathrm{SU}(1,n)$. We use the already introduced technique of reduction to Heisenberg groups. So let $Z_1 \in \mathfrak{z}$ be a normalized element and $\mathfrak{n}_1 = \mathfrak{n}/\mathfrak{z}_1^\perp$ as before. We let S_1 be the harmonic group associated to N_1 and note that there is a natural group homomorphism $S \to S_1$ which extends to a holomorphic map $S_\mathbb{C} \to (S_1)_\mathbb{C}$ which maps $\mathrm{Cr}(S)$ onto $\mathrm{Cr}(S_1)$. Now the assertion is true for $\mathrm{Cr}(S_1)$ in view of [9] (proof of Th. 3.2) and Theorem 3.4. Since (4.1) is true for S if it is true for all choices of S_1, the assertion follows. □

As a consequence of this fact, we obtain as in [9] that:

Theorem 4.2. *Every $\mathcal{L}$-eigenfunction on S extends to a holomorphic function on* $\mathrm{Cr}(S)$.

Proof. (analogous to the proof of Th. 3.2 of [9]). Let f be an $\mathcal{L}$-eigenfunction on S. As $\mathcal{L}$ is elliptic, the regularity theorem for elliptic differential operators implies that f is an analytic function. Hence f extends to some holomorphic function in a neighborhood of S in $S_\mathbb{C}$.

As $\mathcal{L}$ is S-invariant, we may assume that this neighborhood is S invariant. Let $0 \leq t \leq 1$ and define $\mathcal{D}_t := t\mathcal{D}$ and correspondingly D_t. We have shown that f extends holomorphically to a domain $SD_t \subset S_\mathbb{C}$ for some $0 < t \leq 1$.

If $t = 1$, we are finished. Otherwise we find a $(Y, r) \in \partial\mathcal{D}_t$ such that f does not extend beyond $z = s\exp(irH)\exp(iY)$ for some $s \in S$. By S-invariance we may assume that $s = 1$. Set $z := \exp(irH)\exp(iY)$. By our previous Proposition $\mathcal{L}_{Sz}$ is elliptic. Now it comes down to choose appropriate local coordinates to see that f extends holomorphically on a complex cone based at z. One has to verify condition (9.4.16) in [6], Cor. 9.4.9 so that [6], Cor. 9.4.9, applies and one concludes that f is holomorphic near z – see [9], p. 837-838, for the details. This is a contradiction and the theorem follows. □

4.2. **Maximality of** $\mathrm{Cr}(S)$

We begin with a collection of some facts about Poisson kernels on S. Let us denote by $s : S \to S$ the geodesic symmetry, centered at the identity. Every $z \in S_\mathbb{C}$ can be uniquely written as $z = n(z)a(z)$ with $n \in N_\mathbb{C}$ and $a \in A_\mathbb{C}$. As $\mathrm{Cr}(S)$ is simply connected, we obtain for every $\lambda \in \mathfrak{a}_\mathbb{C}^*$ a holomorphic map

$$a^\lambda : \mathrm{Cr}(S) \to \mathbb{C}, \ z \mapsto e^{\lambda \log a(z)} .$$

The function $P_\lambda := a^\lambda \circ s$ on S is referred to as *Poisson kernel on S with parameter λ*. We note that both a^λ and P_λ are $\mathcal{L}$-eigenfunctions [3, 1]. We shall say in the sequel that $\lambda \in \mathfrak{a}_\mathbb{C}^*$, resp. P_λ, is *positive* if $\mathrm{Re}\,\lambda$ is a positive multiple of the element $\beta \in \mathfrak{a}_\mathbb{C}^*$ defined by $\beta(H) = 1$.

Theorem 4.3. *Let $\lambda \in \mathfrak{a}_\mathbb{C}^*$ be positive. Then $\mathrm{Cr}(S)$ is the largest S-invariant domain in $S_\mathbb{C}$ containing S to which P_λ extends holomorphically.*

Proof. First the theorem is true for symmetric S. To be more precise, given $z \in \partial\,\mathrm{Cr}(S)$, then it was shown in [10], Th. 5.1 (and corrigendum in [8], Remark 4.8) that there exists an $s \in S$ such that the basic spherical function ϕ_0 on $S \simeq X$ does not extend beyond sz. In view of the integral formulas for holomorphically extended spherical functions (see [11], Th. 4.2) it follows that $\mathrm{Cr}(S)$ is the maximal S-invariant domain in $\mathrm{Cr}(S)$ to which any positive P_λ extends holomorphically.

We apply $\mathrm{SU}(2,1)$-reduction (see [2], section 2) and the fact that Poisson-kernels "restrict", i.e. if $z \in \mathrm{Cr}(S)$, then we can put $z \in \mathrm{Cr}(S_1)$ with $\mathrm{Cr}(S_1)$ an $\mathrm{SU}(2,1)$-crown contained in $\mathrm{Cr}(S)$ such that the restriction of the Poisson kernel $P_\lambda := P_\lambda|_{S_1}$ is a positive Poisson kernel on S_1. This is clear from the explicit formula for P_λ (see [1], formula (2.35)). Thus the situation is reduced to the symmetric case and the theorem proved. □

Corollary 4.4. *$\mathrm{Cr}(S)$ is the largest S-invariant domain in $S_\mathbb{C}$ which contains S with the property that every $\mathcal{L}$-eigenfunction on S extends to a holomorphic function on $\mathrm{Cr}(S)$.*

Corollary 4.5. *The geodesic symmetry extends to a holomorphic involutive map $s : \mathrm{Cr}(S) \to \mathrm{Cr}(S)$.*

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